Eigenvalues of the Laplacian

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In today's lecture we will see yet another application of layer potential techniques: Eigenvalues of the Laplacian. To this end, let Ω be a C^0 domain and consider

$$\begin{cases} (k^2 + \Delta) u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{in } \partial\Omega \end{cases}$$

The values k^2 for which a nontrivial solution exists are precisely the Neumann eigenvalues for the Laplacian. To make use of layer potential techniques, recall that any solution u to the above admits the representation

$$u(x) = S_\Omega^{k^2}\left[\frac{\partial u}{\partial n}\Big|_-\right](x) + D_\Omega^{k^2}[u](x) \qquad \text{in } D$$

Applying the Neumann boundary condition then yields

$$u(x) = D_\Omega^{k^2}[u](x) \qquad \text{in } D$$

Using the jump relations, we obtain that on the boundary it must hold that

$$\begin{aligned} u(x)|_- &= \tfrac{1}{2} u(x)|_- + K_\Omega^{k^2}[u|_-](x) && \text{in } \partial D \\ \Leftrightarrow \quad 0 &= -\tfrac{1}{2} u(x)|_- + K_\Omega^{k^2}[u|_-](x) && \text{in } \partial D \end{aligned}$$

i.e. $u|_-$ is an eigenfunction of $K_\Omega^{k^2}$ associated to $\frac{1}{2}$. To properly justify this claim, we need to show that $u|_-$ does not identically vanish on $\partial\Omega$.

Thus suppose u vanishes on $\Gamma \subset \partial\Omega$. Let B be a small ball s.t. $B \cap \partial\Omega \subset \Gamma$. We define

$$\tilde{u}(x) = \begin{cases} u(x), & x \in B \cap \Omega \\ 0 & \text{else.} \end{cases}$$

Then $\tilde{u} \in H^1(B)$ and for any $\varphi \in C_c^\infty$

$$\int_B \nabla \tilde{u} \cdot \nabla \bar{\varphi} - k^2 \tilde{u} \bar{\varphi} \, dx = 0$$

Elliptic regularity implies that it is analytic in B. But this forces $\tilde{u}$ to vanish in all of B by the identity principle: Since $\tilde{u}$ vanishes on Γ, for $x \in \Gamma$ all tangential derivatives vanish. As it is a Neumann mode, its normal derivative also vanishes. Iterating this argument, we find that all derivatives of $\tilde{u}$ must vanish, hence $\tilde{u}$ vanishes identically

There is also a simpler trick, using Rellich's identity: Testing with $x \cdot \nabla u$, we have

$$\begin{aligned} 0 &= \int (\Delta + k^2) u \, \overline{(x \cdot \nabla u)} \\ &= \int k^2 u \, \overline{(x \cdot \nabla u)} - |\nabla u|^2 - \nabla u \cdot x \, \overline{\Delta u} \, dx \\ &= 2 \int i k^2 \, \mathrm{Im}\left(u \, \overline{(x \cdot \nabla u)} \right) - |\nabla u|^2 \, dx \\ &\Rightarrow \|\nabla u\|^2 = 0 \end{aligned}$$

which together with $u|_{\partial\Omega} = 0$ implies $u = 0$

One can further show the following proposition

Proposition $-1/2 + K_\Omega^w$ is invertible for all but a discrete subset of $\mathbb{C}$. Moreover, $w \to (-1/2 + K_\Omega^w)^{-1}$ is a meromorphic, operator valued function.

On Monday, we have seen how we can solve this problem numerically. To validate our numerical methods, it is always important to have some explicit examples available.

We therefore focus on the case $\Omega = B_1(0) \subset \mathbb{R}^d$.
In spherical coordinates, the Laplacian has the form

$$\Delta u = (d-1)\, r^{-1} \frac{\partial u}{\partial r} + \frac{\partial u}{\partial r^2} + r^{-2} \Delta_{S^{d-1}} u$$

where $\Delta_{S^{d-1}}$ is the Laplace-Beltrami operator. It thus becomes natural to look for solutions of the form

$$u(x) = U(kr) V(\omega)$$

Separation of variables yields

$$(U'' + (d-1) t^{-1} U' + U) V + t^{-2} U \Delta_{S^{d-1}} V = 0$$

where $t = kr$. We obtain two equations:

i) $U'' + (d-1) t^{-1} U' + (1 - \lambda^2 t^{-2}) U = 0$
ii) $\Delta_{S^{d-1}} V = -\lambda^2 V$

i) is known as the Bessel equation, while ii) is an eigenvalue problem for the spherical Laplacian

Bessel functions:
Bessel's equation is commonly written as

(B) $$t^2 f'' + t f' + (t^2 - \nu^2) f = 0$$

To see that this is equivalent to i), write $f = t^{-\alpha} U$ for some α yet to be decided. This yields

$$0 = \alpha(\alpha-1) t^{\alpha-2} f + 2\alpha t^{\alpha-1} f' + t^\alpha f'' + (d-1)\alpha t^{\alpha-2} f + (d-1) t^{\alpha-1} f' + (1 - \tfrac{\lambda^2}{t^2}) t^\alpha f$$

$$= t^\alpha f'' + [2\alpha t^{\alpha-1} + (d-1) t^{\alpha-1}] f' + [\alpha(\alpha-1) t^{\alpha-2} + (d-1)\alpha t^{\alpha-2} + (1 - \lambda^2/t^2) t^\alpha] f$$

$$= t^{\alpha-2}\left[t^2 f'' + t\left[2\alpha + (d-1)\right] f' + \left[\alpha(\alpha-1) + (d-1)\alpha + t^2 - \lambda^2\right] f\right]$$

Set $2\alpha + (d-1) = 1 \iff \alpha = 1 - d/2$ to obtain

$$t^{-1-d/2}\left[t^2 f'' + t f' + (t^2 - \nu^2) f\right] = 0$$

with $\nu = \alpha^2 + \lambda^2$.

How do we solve this equation? Power-series ansatz:

$$f(x) = \sum a_n x^{n+s}$$

Proposition: Consider the Bessel functions of the first kind

$$J_\nu(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu}$$

They solve (B).
Proof: Exercise!

Since (B) is a second order ODE, there exists a second solution

Proposition: The Bessel function of the second kind is given by

$$Y_\nu(x) := \frac{\cos(\nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

it is also a solution to (B)
Proof: Exercise!

Remark: For integral values of ν, Y_ν can still be defined as limit for $x \neq 0$.

We thus can write the general solution to Bessel's equation

$$f(x) = A J_\nu(x) + B Y_\nu(x)$$

Proposition: For $r \geq 0$, the following hold

i) J_r is smooth at the origin
ii) Y_r has a singularity of order $O(x^{-r})$ at the origin if $r > 0$ and $O(\log x)$ if $r = 0$
iii) As $|x| \to \infty$, the following asymptotics hold:

$$J_r(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi r}{2} - \frac{\pi}{4}\right)$$
$$Y_r(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi r}{2} - \frac{\pi}{4}\right)$$

In the literature, there exist many kinds of Bessel functions. For our purposes, the following will be important

i) spherical Bessel functions of the first and second kind

$$j_m(x,d) := \sqrt{\frac{\pi}{2}} \frac{J_{m+d/2-1}(x)}{x^{d/2-1}} \quad \text{and} \quad y_m(x,d) := \sqrt{\frac{\pi}{2}} \frac{Y_{m+d/2-1}(x)}{x^{d/2-1}}$$

ii) Hankel functions of the first and second kind

$$H_r^{(1)} := J_r + i Y_r \quad \text{and} \quad H_r^{(2)} = J_r - i Y_r$$

iii) spherical Hankel functions of the first and second kind

$$h_m^{(1)}(x,d) = j_m(x,d) + i y_m(x,d) \quad \text{and} \quad h_m^{(2)}(x,d) = j_m(x,d) - i y_m(x,d)$$

The Hankel functions are of particular interest to us due to their asymptotics

$$h_m^{(2)}(kr,d) \sim \frac{1}{(kr)^{(d-1)/2}} \exp\left(i\left(kr - \tfrac{1}{2}\pi m + \frac{d-1}{4}\pi\right)\right)$$

$\Rightarrow$ This satisfies the Sommerfeld radiation condition. In particular, the fundamental solution is given by

$$G^k(r) := \frac{k^{d-2}}{2(2\pi)^{(d-1)/2}} i h_0^{(1)}(d, kr)$$

Consistency check:
We know from the lecture that the Green's function for $d=3$ is given by

$$G^k(x) = \frac{\exp(ikr)}{4\pi r}$$

It can be shown that for d odd satisfy

$$h_0^{(2)}(r,d) = (-i)^{\frac{d-1}{2}} \frac{e^{ir}}{r} \sum_{m=0}^{\frac{d-3}{2}} \frac{i^m}{m!(2x)^m} \frac{(\frac{d-2}{2}+m)!}{(\frac{d-2}{2}-m)!}$$

For $d=3$ this yields

$$h_0^{(2)}(kr,3) = -i \frac{e^{ikr}}{kr}$$

Thus

$$\frac{k^{3-2}}{2(2\pi)^{(3-3)/2}} \cdot h_0^{(2)}(kr,d) = \frac{e^{ikr}}{4\pi r}$$

Spherical Harmonics

We now turn to ii), that is solutions of

$$-\Delta_{S^{d-1}} V = \lambda V$$

Denote $\mathbb{P}_d^\ell$ the space of complex valued polynomials of total degree less than ℓ.

Theorem (Stars and Bars):
Let $n, k \in \mathbb{N}$. The number of $\mathbb{N} \cup \{0\}$-valued k-tuples whose sum is n is equal to $\binom{n+k-1}{k-1}$

Proof: The number of such k-tuples is equal to the number of ways one can separate n stones using $k-1$ sticks. This in turn is equivalent to the number of strings of size $n+k-1$ containing n zeros and $k-1$ ones. Out this is easily seen to be equal to $\binom{n+k-1}{k-1}$.

Denote by $\tilde{\mathbb{P}}_d^\ell$ the space of homogenous polynomials of degree ℓ in d variables, using the previous theorem we easily find

Corollary: $\dim \tilde{\mathbb{P}}_d^\ell = \binom{\ell+d-1}{d-1}$

This leads to

Theorem: $\dim \mathbb{P}_d^\ell = \binom{\ell+d}{d}$

Proof: It is easy to see that

$$\begin{aligned}\dim \mathbb{P}_d^\ell &= \dim \tilde{\mathbb{P}}_d^\ell + \dim \mathbb{P}_d^{\ell-1} \\ &= \sum_{k=0}^{\ell} \dim \tilde{\mathbb{P}}_d^k \\ &= \sum_{k=0}^{\ell} \binom{k+d-1}{d+1} \\ &= \binom{\ell+d}{d}\end{aligned}$$

Finally, we define $\mathbb{H}_d^\ell$ the space of homogenous polynomials of degree ℓ in d variables that are harmonic. That is, $p \in \mathbb{H}_d^\ell$ satisfies

$$\Delta p = 0$$

Lemma: The space $\tilde{\mathbb{P}}_d^\ell$ is the direct sum of the space $\mathbb{H}_d^\ell$ and $|x|^2 \tilde{\mathbb{P}}_d^{\ell-2}$. That is for every $p \in \tilde{\mathbb{P}}_d^\ell$ there exist $p_1 \in \mathbb{H}_d^\ell$, $p_2 \in \mathbb{P}_d^{\ell-2}$ s.t.

$$p = p_1 + |x|^2 p_2$$

Proof: Define the scalar product

$$\langle \cdot, \cdot \rangle_\ell : \mathbb{P}_d^\ell \times \mathbb{P}_d^\ell \to \mathbb{R}, \quad (p, q) \longrightarrow p\left(\frac{\partial}{\partial x}\right) \bar{q} \Big|_{x=0}$$

where we understand $\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}\right)$. We then have

$$\begin{aligned} 0 &= \langle p, |x|^2 p_2 \rangle \\ &= \overline{\langle p_2 |x|^2, p \rangle} \\ &= \overline{p_2}\left(\frac{\partial}{\partial x}\right) \Delta p \\ &= \langle p_2, \Delta p \rangle \end{aligned}$$

$\langle \cdot, \cdot \rangle$ is essentially just a weighted scalar product on $\mathbb{R}^{\dim(\mathcal{P}_d^\ell)}$

Hence the orthogonal of $|x|^2 \mathcal{P}_d^{\ell-2}$ is indeed $\mathcal{H}_d^\ell$.

Lemma: The restriction of $\tilde{\mathcal{P}}_d^\ell$ to the unit sphere decomposes as

$$\begin{aligned} \tilde{\mathcal{P}}_d^\ell &:= \begin{cases} Y_0 \oplus Y_2 \oplus \dots \oplus Y_\ell & \text{if } \ell \text{ is even} \\ Y_1 \oplus Y_3 \oplus \dots \oplus Y_\ell & \text{if } \ell \text{ is odd} \end{cases} \end{aligned}$$

where Y_ℓ is the restriction of $\mathcal{H}_d^\ell$ to the sphere S^{d-1}

Proof: Immediate consequence of the previous lemma.

To characterise the harmonic, homogenous polynomials as eigenfunctions for the spherical laplacian, we need the following lemma

Lemma: Let $y \in C^2(S^{d-1})$ and $\nu \in \mathbb{R}$. We have

$$\Delta\left(r^\nu y(w)\right) = 0 \iff -\Delta_{S^{d-1}} y = \ell(\ell + d - 2) y.$$

Proof:

$$\begin{aligned} 0 &= \Delta\left(r^\nu y(w)\right) \\ &= (d-1)\, \nu\, r^{\nu-2} y(w) + \nu(\nu-1)\, r^{\nu-2} y(w) + r^{\nu-2} \Delta_{S^{d-1}} y(w) \end{aligned}$$

$$\implies \quad \nu(\nu + d - 2) y(w) = -\Delta_{S^{d-1}} y(w)$$

This now allows us to prove the desired theorem

Theorem: $L^2(S^{d-1})$ decomposes as

$$L^2(S^{d-1}) = \bigoplus_{\ell=0}^{\infty} Y_\ell$$

and the spaces Y_ℓ are orthogonal. Denote by Y_ℓ^m an orthonormal basis of Y_ℓ. We have

$$-\Delta_{S^{d-1}} Y_\ell^m = \ell(\ell+d-2)\, Y_\ell^m$$

Hence these functions form an orthonormal basis of eigenfunctions. In particular, the dimension of the eigenspace associated to $\ell(\ell+d-2)$ is $\dim Y_\ell \cdot \text{bin}(\ell+d-2, d-2)$.

Proof: From Stone-Weierstrass (Weierstrass approximation Theorem) we know that the set of polynomials is dense in $L^2(S^{d-1})$. Hence indeed

$$L^2(S^{d-1}) = \bigoplus_{\ell=0}^{\infty} Y_\ell$$

The previous lemma implies that each function $y_\ell \in Y_\ell$ satisfies

$$-\Delta_{S^{d-1}} y_\ell = \ell(\ell+d-2)$$

As eigenfunctions to different eigenvalues of a self-adjoint operator must be orthogonal, the spaces Y_ℓ are indeed orthogonal to each other. Next,

$$\begin{aligned}\dim Y_\ell &= \dim \tilde{P}_d^\ell - \dim \tilde{P}_d^{\ell-2} \\ &= \text{bin}(\ell+d-1, d-1) - \text{bin}(\ell+d-2, d-1) \\ &= \text{bin}(\ell+d-2, d-2)\end{aligned}$$

and the claim follows by choosing for each Y_ℓ an orthonormal basis Y_ℓ^m.

Far-Field expansion

We now want to use our newly gained knowledge to understand the far-field behaviour of solutions to the Helmholtz problem. To this end, we expand the fundamental solution in terms of spherical harmonics

$$G(x-rw) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{\dim Y_\ell} Y_\ell^m(w) \int_{S^{d-1}} G(x-r\eta)\, \overline{Y_\ell^m}(\eta)\, d\eta$$

To compute the integrals

$$\int_{S^{d-1}} G(x-r\eta)\, \overline{Y_\ell^m(\eta)}\, d\eta$$

we will make use of the following two identities:

$$\Delta(r^\ell Y_\ell^m) = 0 \text{ in } \mathbb{R}^d \quad \text{and} \quad \Delta(r^{-(\ell+d-2)} Y_\ell^m) = 0 \text{ in } \mathbb{R}^d \setminus \{0\}$$

which are the consequence of one of the preceding lemmas.
Define

$$u(x) = |x|^\ell Y_\ell^m(x/|x|) \quad \text{and} \quad v(x) = |x|^{-(\ell+d-2)} Y_\ell^m(x/|x|)$$

We use Green's second identity to obtain for $x \in \mathbb{R}^d \setminus D_R$

$$0 = \int_{D_R} \Delta u(y) G(x-y)\, dy = \int_{\partial B_R} G(x-y) \frac{\partial}{\partial n} u - u \frac{\partial}{\partial n} G(x-y)\, dy$$

$$0 = \int_{\mathbb{R}^d \setminus D_R} \Delta v(y) G(x-y)\, dy - \int_{\partial D_R} v \frac{\partial}{\partial n} G(x-y) - G(x-y) \frac{\partial}{\partial n} v\, dy + v(x)$$

Green's second:

$$\int \psi \nabla^2 \varphi - \varphi \nabla^2 \psi = \int_{\partial} \psi \frac{\partial \varphi}{\partial n} - \frac{\partial \psi}{\partial n}$$

Using $u = |x|^{2\ell+d-2} v$ we obtain

$$0 = \int_{\partial B_R} G(x-y) \left(\frac{\partial}{\partial n} u - R^{2\ell+d-2} \frac{\partial}{\partial n} v \right) dy + R^{2\ell+d-2} v(x)$$

Since $\frac{\partial}{\partial n} u = \ell r^{-1} u - \ell r^{2\ell+d-3} v = \frac{\ell}{-(\ell+d-2)} \frac{\partial}{\partial n} v$

We find

$$R^{2\ell+d-2} v + \int_{\partial B_R} G(x-y) \left(\frac{\partial}{\partial n} u - R^{2\ell+d-2} \frac{\partial n}{\partial r} v \right) Y_\ell^m \, dy = 0$$

That is for $\ell \neq 0$:

$$- \frac{R^\ell}{2\ell+d-2} |x|^{-(\ell+d-2)} \overline{Y_\ell^m(x/|x|)} = \int_{\partial B_R} G(x-y) \overline{Y_\ell^m} \, dy$$

Whereas for $\ell = 0$ one can compute explicitely

$$\frac{1}{|\partial B_R|} \int_{B_R} G(x-y) \, dy = G(x).$$

This yields

Proposition: The following expansion holds for $d \neq 2$

$$G(x-y) = G(x) - \sum_{\ell=1}^{\infty} \sum_{m=1}^{\dim Y_\ell} \frac{|x|^{-(\ell+d-2)} |y|^\ell}{2\ell+d-2} \overline{Y_\ell^m(x/|x|)} \, Y_\ell^m(y/|y|)$$

On the other hand, Taylor expansion yields

$$G(x-y) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!} \langle \nabla^m G(x), y \rangle \quad \text{for } |x| > |y|.$$

Upon comparing coefficients, we find

$$\langle \nabla^m G(w), \eta \rangle = (-1)^{m+1} \frac{\ell!}{2\ell+d-2} \sum_{m=1}^{\dim Y_\ell} \overline{Y_\ell^m(w)} \, Y_\ell^m(\eta)$$

A similar expansion exists for solution of the Helmholtz equation.

Further Reading & Sources

i) Bessel Functions and their Applications, Jennifer Niedzielu, 2008
ii) Lecture n°3 - Bessel functions and spherical harmonics, Florian Feppon, 2021
iii) Acoustic and Electromagnetic Equations, Jean-Claude Nédélec
iv) Tutorial notes, Chapter 2