BEM & Operator Approximation

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We are again interested in solutions of the Helmholtz Problem

$$\begin{cases} (\Delta + k^2) u = 0 & \text{in } D \\ (\Delta - k_0^2) u = 0 & \text{in } \mathbb{R}^d \setminus \overline{D} \\ u_+ = u_- & \text{in } \partial D \\ \rho^{-1} \partial_n u|_+ - \rho_0^{-1} \partial_n u|_- & \text{in } \partial D \\ \text{Sommerfeld radiation condition} \end{cases}$$

You already know that the solutions of this problem can be described using layer potentials

$$S_D^k[\varphi] = \int_{\partial D} G^k(x,y)\, \varphi(y)\, dy$$

$$D_D^k[\varphi] = \int_{\partial D} \frac{\partial}{\partial n_y} G^k(x,y)\, \varphi(y)\, dy$$

$$(K_D^k)'[\varphi] = \int_{\partial D} \frac{\partial}{\partial n_x} G^k(x,y)\, \varphi(y)\, dy$$

The following jump relations hold

$$\frac{\partial}{\partial n} S_D^k[\varphi]\Big|_\pm = (\pm \tfrac{1}{2} \mathrm{Id} + (K_D^k)')[\varphi]$$

$$D_D^k[\varphi]\Big|_\mp = (\mp \tfrac{1}{2} \mathrm{Id} + K_D^k)[\varphi]$$

It can be shown that a solution u admits the representation

$$u(x) = D_D^k[u](x) - S_D^k\left[\frac{\partial u}{\partial n}\right](x)$$

However, since both S_D^k and D_D^k solve the Helmholtz equation, we could also choose to represent it solely using S_D^k:

$$u(x) = \begin{cases} S_D^k[\varphi](x) & \text{in } D \\ S_D^{k_0}[\psi](x) & \text{in } \mathbb{R}^3 \setminus D \end{cases}$$

With some unknown densities $\varphi, \psi \in L^2(\partial D)$. These densities must satisfy

i) Continuity: $S_D^k[\varphi](x) - S_D^{k_0}[\psi](x) = 0, \quad x \in \partial D$

ii) Flux Continuity: $\frac{\partial}{\partial n} S_D^{k_0}[\psi](x)\Big|_- = \delta \frac{\partial}{\partial n} S_D^k[\varphi](x)\Big|_+, \quad x \in \partial D$

where $\delta = \rho_0/\rho$. Using the jump relations, ii) can be written as

$$(-\tfrac{1}{2} + (K_D^{k_0})')[\psi] - \delta(\tfrac{1}{2} + (K_D^k)')[\varphi] = 0$$

If we define the operator

$$A(k,\delta) = \begin{pmatrix} S_D^k & -S_D^{k_0} \\ -\delta(\tfrac{1}{2} + (K_D^k)') & (-\tfrac{1}{2} + (K_D^{k_0})') \end{pmatrix}$$

Then we need to find non-trivial solutions $(\varphi, \psi) \in L^2(\partial D)$ to

$$A(k,\delta)\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The resonant frequencies k that admit such non-trivial solutions are called characteristic values.

In our group, where particularly interested in the sub wave length resonances satisfying

$$k(\delta) \longrightarrow 0 \quad \text{as} \quad \delta \longrightarrow 0$$

Operator approximations for Fredholm integral equations

These kinds of integral equations are known as Fredholm integral equations of the...

i) first kind: $f(x) = \int_{\partial D} K(x,y)\,\varphi(y)\,dy$

ii) second kind: $f(x) = \varphi(x) - \int_{\partial D} K(x,y)\,\varphi(y)\,dy$

For the sake of simplicity, we will assume the kernel K to be continuous. In numerically solving such equations we need to consider two things

i) How do we discretise $\longrightarrow$ Next week

ii) Do our approximations converge in any meaningful sense?

Today we will concern ourselves with ii). To this end, we will need some tools from functional analysis.

Definition: A Banach space is a complete normed space.

Definition: Let X, Y be Banach spaces. A bounded linear operator is a map $A: X \longrightarrow Y$ s.t.

$$\|A\|_{L(X,Y)} := \sup_{\varphi \in X \setminus \{0\}} \frac{\|A\varphi\|_Y}{\|x\|_X} < \infty$$

The set of bounded linear maps is denoted $B(X,Y)$ and is itself a Banach space.

We now consider a sequence of operators $A_1, \{A_n\}_{n \in \mathbb{N}} \in B(X,Y)$. Moreover let $f \in Y$ and $f_n \in Y$ a sequence converging to f. We now replace the equation

$$A\varphi = f \qquad \text{with} \qquad A_n \varphi_n = f_n$$

and ask ourselves about the convergence of φ_n towards φ.

Definition (Pointwise / Strong Convergence): We say A_n converges to A pointwise / strongly, if for every $\varphi \in X$

$$\lim_n \| A_n \varphi - A\varphi \|_X = 0$$

Definition (Uniform / Norm Convergence): We say A_n converges to A uniformly / in norm, if

$$\lim_n \| A_n - A \|_{L(X,Y)} = 0$$

Example: $\ell^2(\mathbb{N})$, let P_n denote the projection onto the first n coordinates. Then P_n converges to Id strongly, since

$$\| (P_n - Id)\varphi \|^2 = \sum_{k=n+1}^{\infty} \varphi_k^2 \longrightarrow 0$$

for $\varphi \in \ell^2(\mathbb{N})$. However, the sequence does not converge uniformly:

$$\| (P_n - Id) e_{n+1} \| = 1 \qquad \forall n \in \mathbb{N}.$$

The following theorem will prove useful to us.

Theorem (Neumann Series): Let $A \in B(X,X)$ and suppose $\|A\| < 1$. Then the operator $(Id - A)$ is invertible and its inverse is given by

$$(Id - A)^{-1} = \sum_{k=0}^{\infty} A^k$$

and satisfies

$$\| (Id - A)^{-1} \| \leq (1 - \|A\|)^{-1}$$

Proof: First, let $\varphi \in X$ and note that

$$\left\| \sum_{k=0}^{\infty} A^k \varphi \right\| \leq \|\varphi\| \sum_{k=0}^{\infty} \|A\|^k \leq \frac{\|\varphi\|}{1-\|A\|}$$

Hence the map $\varphi \longrightarrow \sum_{k=0}^{\infty} A^k \varphi$ is well defined and bounded. Note further that

$$(1-A)\sum_{k=0}^{\infty} A^k = \sum_{k=0}^{\infty} A^k - A^{k+1} = Id$$

Since the above is a telescopic sum.

Theorem: Let X, Y be Banach spaces and $A, \{A_n\}_{n\in\mathbb{N}} \in B(X,Y)$. Assume $A_n \to A$ in norm and that A is invertible. Then for n s.t.

$$\|A^{-1}(A_n - A)\| < 1$$

the inverse operator exist and are bounded by

$$\|A_n^{-1}\| < \frac{\|A^{-1}\|}{1-\|A^{-1}(A_n-A)\|}$$

For the solutions of the equations

$$A\varphi = f \quad \text{and} \quad A_n \varphi_n = f_n$$

we have the estimate

$$\|\varphi - \varphi_n\| < C\left(\|(A_n - A)\varphi\| + \|f_n - f\|\right)$$

Proof: Suppose $\|A^{-1}(A-A_n)\| < 1$. We have

$$\begin{aligned} A_n &= A - (A - A_n) \\ &= A\left(Id - A^{-1}(A - A_n)\right) \end{aligned}$$

and the claim (as well as the bound) follow from the Neumann series theorem. For the upper bound on $\|\varphi_n - \varphi\|$ note that

$$\begin{aligned}\varphi - \varphi_n &= A_n^{-1} A_n(\varphi - \varphi_n) \\ &= A_n^{-1}\left((A_n - A)\varphi + A\varphi - A_n\varphi_n \right) \\ &= A_n^{-1}\left((A_n - A)\varphi + f - f_n \right)\end{aligned}$$

This tells us something about the convergence of Fredholm integral equations of the first kind. For the second kind we have to work a bit harder

Definition: A set $\mathcal{A} \subset B(X,Y)$ is called collectively compact, if for each bounded set $U \subset X$ the image $\mathcal{A}(U) \subset Y$ is relatively compact.

Theorem: Let $A, \{A_n\}_n \in B(X,Y)$ and suppose $\{A_n\}_n$ is collectively compact and $A_n \to A$ pointwise. Then

$$\|(A - A_n)A\|, \|(A - A_n)A_n\| \longrightarrow 0$$

Proof: We first note that A is compact. To see this, let $U \subset X$. Then $\{A_n\}_n(U)$ is relatively compact. On the other hand

$$\overline{\{A_n\}_n(U)} \supset \overline{A(U)}$$

Hence $\|(A - A_n)A\| \to 0$ is a standard result of pointwise convergence on a compact set. To see the second convergence remember that $\{A_n\}_n(U)$ is compact.

To arrive at our main result, we will need one more result from functional analysis

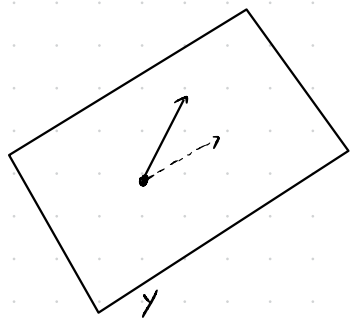
Theorem (Fredholm alternative): Let $A \in B(X,X)$ be compact. Then $Id - A$ is injective if and only if it is surjective.

We will need another lemma to prove this

Lemma (Riesz): Let Y be a proper closed subspace of a normed vector space X. For every $\varepsilon > 0$, there exists some $x \in X$ s.t. $\|x\| = 1$ and

$$\|x - y\| \geq 1 - \varepsilon \qquad \forall\, y \in Y.$$

Proof:



Let $x \in X \setminus Y$ with $\|x\| = 1$. Let

$$\alpha = \operatorname{dist}(x, Y) > 0$$

If $\alpha \geq 1 - \varepsilon$, we are done. Otherwise, choose y' s.t.

$$\|x - y'\| < \alpha(1 + \varepsilon)$$

and set $x' = (x - y')\|x - y'\|^{-1}$. We have

$$\|x' - y\| = \frac{1}{\|x - y'\|}\,\|x - y' - \|x - y'\| y\|$$

$$\geq \frac{1}{\alpha(1+\varepsilon)} \operatorname{dist}(x, Y)$$

$$= \frac{\alpha}{\alpha(1+\varepsilon)}$$

$$= \frac{1}{1+\varepsilon} > 1 - \varepsilon$$

Proof (of Fredholm's alternative): The desired statement is equivalent to

$$Id - A \text{ is injective} \iff (Id - A)^{-1} \text{ is well defined and bounded.}$$

Clearly, "$\Leftarrow$" holds. We proceed with the converse direction. If $Id - A$ is not invertible, one of the following must hold

i) not injective
ii) non-closed image and injective
iii) closed but non-dense image and injective

In case i) we are done. Thus suppose ii) applies. Then there exists

$$y \in \overline{\mathrm{Im}(\mathrm{Id}-A)} \setminus \mathrm{Im}(\mathrm{Id}-A).$$

Let x_n be s.t. $(\mathrm{Id}-A)x_n \to y$. Suppose x_n is unbounded, then set $\tilde{x}_n = x_n/\|x_n\|$. Then

$$\lim (\mathrm{Id}-A)\tilde{x}_n = \lim (\mathrm{Id}-A)x_n/\|x_n\| = 0$$

contradicting our assumption that $(\mathrm{Id}-A)$ is injective. Thus, since A is compact, we may define $z = \lim Ax_n$. Note that

$$\begin{aligned} 0 &= \lim \|y - (\mathrm{Id}-A)x_n\| \\ &= \lim \|(y+z) - x_n + Ax_n - z\| \\ &\geq \lim \|(y+z) - x_n\| - \|Ax_n - z\| \\ &= \lim \|(y+z) - x_n\| \end{aligned}$$

Thus x_n converges to $y+z$, implying $A(y+z) = y$. Thus $(\mathrm{Id}-A)(y+z) = y$, contradicting our initial assumption.
Finally, we consider iii). Then $\mathrm{Im}(\mathrm{Id}-A)$ is a proper subspace of X and $\mathrm{Id}-A$ is an isomorphism $X \to \mathrm{Im}(\mathrm{Id}-A)$. In particular,

$$X \supsetneq \mathrm{Im}(\mathrm{Id}-A) \supsetneq \mathrm{Im}((\mathrm{Id}-A)^2) \supsetneq \dots$$

Riesz Lemma now implies that there exists $v_m \in \mathrm{Im}((\mathrm{Id}-A)^m)$ s.t.

$$\mathrm{dist}(v_m, \mathrm{Im}((\mathrm{Id}-A)^{m+1})) \geq 1/2.$$

Let $n > m$, then

$$\|A(v_m - v_n)\| = \|(\mathrm{Id}-A)v_n - (\mathrm{Id}-A)v_m - v_n + v_m\|$$

$$= \| [(Id-A)v_n - (Id-A)v_m - v_n] + v_m \|$$

the term in square brackets lies in $\operatorname{Im}((Id-A)^{m+1})$ and hence

$$\| A(v_m - v_n) \| > 1/2$$

But this contradicts compactness of A ↯

This leads us to the convergence theorem for Fredholm integral equations of the second kind.

Theorem: Let $A \in B(X,X)$ and $1-A$ injective. Suppose $\{A_n\} \subset B(X,X)$ is collectively compact and $A_n \to A$ pointwise. Then for n large enough s.t.

$$\| (Id-A)^{-1}(A-A_n) A_n \| < 1$$

The inverse operators $(Id-A_n)^{-1}$ exist and satisfy

$$\| (Id-A_n)^{-1} \| \leq \frac{1 + \| (Id-A)^{-1} A_n \|}{1 - \| (Id-A)^{-1}(A-A_n) A_n \|}$$

For solutions of the equations

$$\varphi - A\varphi = f \qquad \text{and} \qquad \varphi_n - A_n \varphi_n = f_n$$

we have the estimate

$$\| \varphi - \varphi_n \| \leq C \left(\| (A_n - A)\varphi \| + \| f_n - f \| \right)$$

Proof: The Fredholm alternative implies that $Id-A$ is invertible. We have

$$(Id-A)^{-1} = Id + (Id-A)^{-1} A$$

Let

$$B_n = Id + (Id-A)^{-1} A_n$$

Then

$$\begin{aligned} B_n(\mathrm{Id}-A_n) &= \mathrm{Id}-A_n + (\mathrm{Id}-A)^{-1}(\mathrm{Id}-A_n)A_n \\ &= \mathrm{Id} - (\mathrm{Id} - (\mathrm{Id}-A)^{-1}(\mathrm{Id}-A_n))A_n \\ &= \mathrm{Id} - (\mathrm{Id}-A)^{-1}(A_n-A)A_n \end{aligned}$$

Where $S_n := (\mathrm{Id}-A)^{-1}(A_n-A)A_n$ converges to 0 uniformly.

$$B_n(\mathrm{Id}-A_n) = \mathrm{Id}-S_n$$

For n large enough, we hence see that $B_n(\mathrm{Id}-A_n)$, and by extension $(\mathrm{Id}-A_n)$ is injective. Hence $(\mathrm{Id}-A_n)$ is invertible. We obtain

$$(\mathrm{Id}-A_n)^{-1} = (\mathrm{Id}-S_n)^{-1}B_n$$

Which yields

$$\begin{aligned} \|(\mathrm{Id}-A_n)^{-1}\| &\leq \frac{\|B_n\|}{1-\|S_n\|} \\ &\leq \frac{1+\|(\mathrm{Id}-A)^{-1}A_n\|}{1-\|(\mathrm{Id}-A)^{-1}(A_n-A)A_n\|} \end{aligned}$$

To obtain the last bound, note that

$$\begin{aligned} \varphi-\varphi_n &= (\mathrm{Id}-A)^{-1}f - (\mathrm{Id}-A_n)^{-1}f_n \\ &= (\mathrm{Id}-A_n)^{-1}((\mathrm{Id}-A_n)(\mathrm{Id}-A)^{-1}f - f_n \\ &= (\mathrm{Id}-A_n)^{-1}(f-f_n + (A-A_n)(\mathrm{Id}-A)^{-1}f) \\ &= (\mathrm{Id}-A_n)^{-1}(f-f_n + (A-A_n)\varphi) \end{aligned}$$

Further Reading & Sources

i) Tutorial notes section 1.2 and 1.3

ii) Terry Tao's Blog "A proof of the Fredholm alternative"